Beyond Occam's Razor in System Identification: Double-Descent when Modeling Dynamics

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Workshop on Nonlinear System Identification Benchmarks
2021

Neural network performance *vs* size

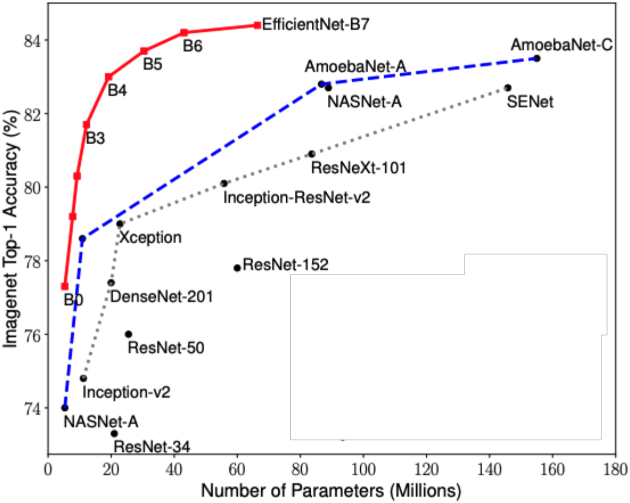
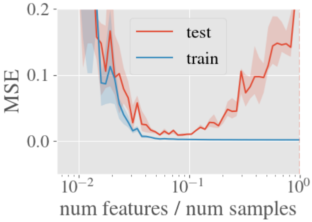


Figure: **Model Size vs. imagenet accuracy.**

Double-descent



(a) U-shaped MSE

Figure: **Perform in CE8 Benchmark.** We show one-step-ahead prediction error in test and training data for a nonlinear ARX model in the CE8 benchmark.

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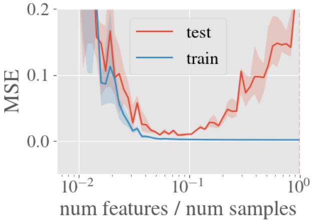

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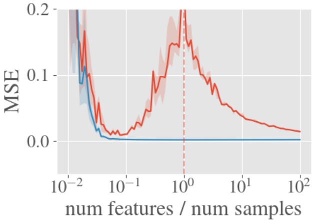

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Related work and historical development

- Random Fourier features, random forest and shallow networks:

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Our contribution

Experimentally show the phenomena in the system identification setting: input-output data from a dynamical system.

Motivation example

$$y_t = \left(0.8 - 0.5e^{-{y_{t-1}}^2}\right) y_{t-1} - \left(0.3 + 0.9e^{-{y_{t-1}}^2}\right) y_{t-2}$$
$$+ u_{t-1} + 0.2u_{t-2} + 0.1u_{t-1}u_{t-2} + v_t,$$
$$v_t \sim \mathcal{N}(0, \sigma_v^2)$$

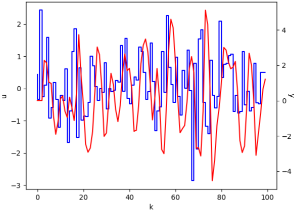

Figure: **System with process noise.** Input in blue and output in red.

Chen, S., Billings, S.A., and Grant, P.M. (1990). *Non-Linear System Identification Using Neural Networks.* International Journal of Control, 51(6), 1191–1214.

Model

Linear-in-the-parameters: Predicted output

$$\hat{y}_t = \theta^{\mathsf{T}} z_t.$$

- $\hat{y}_t \rightsquigarrow$ predicted output

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Nonlinear feature map :

$$z_t = \left(\begin{bmatrix} u_{t-1} \\ u_{t-2} \\ y_{t-1} \\ y_{t-2} \end{bmatrix} \right)$$

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$$z_t = \left(W \begin{bmatrix} u_{t-1} \\ u_{t-2} \\ y_{t-1} \\ y_{t-2} \end{bmatrix} \right)$$

- $W \rightsquigarrow$ Matrix with dimension $m \times 4$

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$$z_t = \sigma \left(W \begin{bmatrix} u_{t-1} \\ u_{t-2} \\ y_{t-1} \\ y_{t-2} \end{bmatrix} \right)$$

- $W \rightsquigarrow$ Matrix with dimension $m \times 4$
- $\sigma \rightsquigarrow$ activation function (element-wise)

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Random matrix: (set in advance)

$$W = \underbrace{\begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,3} \\ w_{2,1} & w_{2,2} & w_{2,3} & w_{2,3} \\ w_{3,1} & w_{3,2} & w_{3,3} & w_{3,3} \\ \vdots & \vdots & \vdots & \vdots \\ w_{m,1} & w_{m,2} & w_{m,3} & w_{m,3} \end{bmatrix}}_{} \left.\vphantom{\begin{bmatrix} w \\ w \\ w \\ \vdots \\ w \end{bmatrix}}\right\} m$$

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Model estimation

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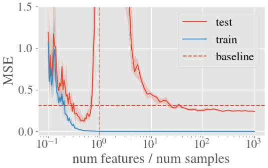
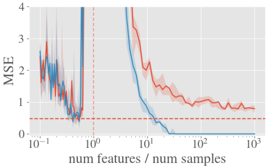
- Overparametrized:

$$\begin{aligned} &\min_{\theta} \|\theta\|_2^2 \\ \text{subject to } & y_t = \theta^\mathsf{T} z_t \\ & \quad \text{for every } t = 1, \cdots, n \end{aligned}$$

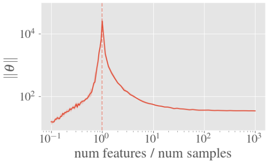
Results



(a) one-step-ahead



(b) free-run simulation



(c) parameter norm

Figure: **Double-descent performance curve.**

Ridge regression

$$\min_{\theta} \sum_{t} (y_i - \theta^\mathsf{T} z_t)^2 + \lambda \|\theta\|^2$$

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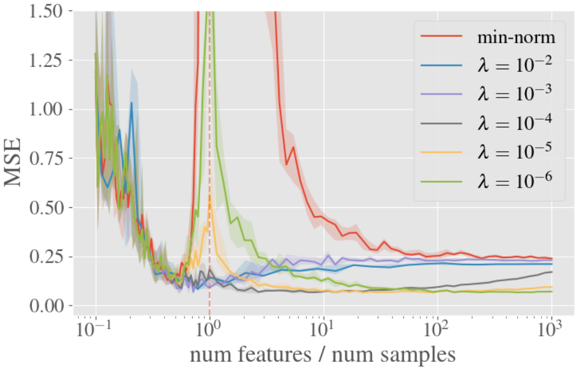

Figure: **Ridge regression with vanishing values of λ.**

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$$y_t = \sum_{i \in \mathcal{S}} \theta_i z_{t,i} \text{ for all } i \in \mathcal{S}$$

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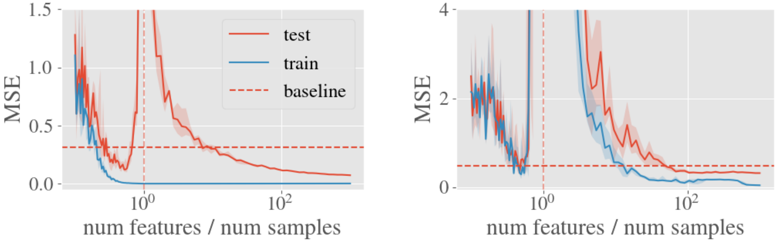


(a) one-step-ahead MSE (b) free-run simulation MSE

Figure: **Ensembles after the interpolation threshold**.

Coupled Electric Drives

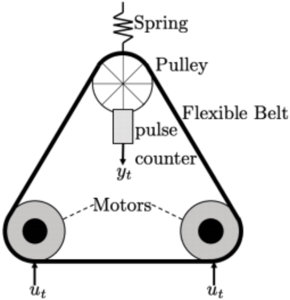
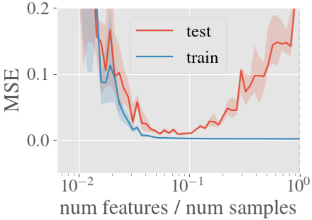


Figure: **Illustration of the CE8 coupled electric drives system**

Wigren, T. and Schoukens, M. (2017). *Coupled electric drives data set and reference models.* Technical Report. Uppsala Universitet, 2017

Double-descent in the CE8 benchmarks



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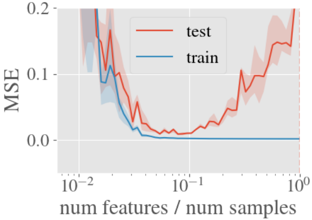
Double-descent in the CE8 benchmarks



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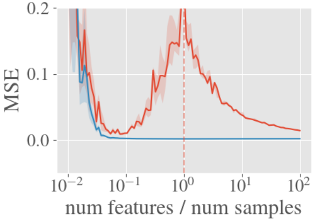

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- Additional experiments: Examples with Random Forest and Radial basis function (RBF) network.
- Studying double descent for *nonlinear ARMAX*, *output error* and other types of models that can handle more general noise types.

Thank you!

To appear in the 19th IFAC Symposium in System Identification.

arXiv.org **Paper:** https://arxiv.org/abs/2012.06341

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Code: https://github.com/antonior92/narx-double-descent

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