Recurrent Structures in System Identification

Antônio H. Ribeiro

Universidade Federal de Minas Gerais - UFMG

Escola de Engenharia

Programa de Pós-Graduação em Engenharia Elétrica - PPGEE

Supervisor: Luis A. Aguirre

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Overview

Introduction

Problem Statement

What is System Identification?

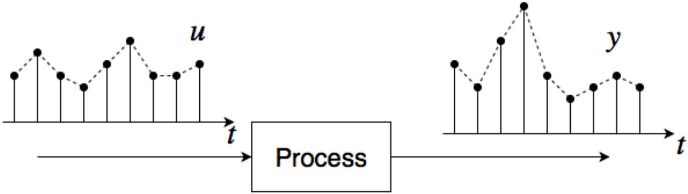

Figure: The system identification problem.

Typical Steps

System Identification Procedure

1. Test design and data collection;
2. Choice of mathematical representation;
   - Dynamic representation;
   - Approximation function;
   - Noise model.
3. Choice of model order and structure;
4. Estimation of model parameters;
5. Model validation.
   - Validation data.
   - One-step-ahead prediction *vs* free-run simulation.

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Nonlinear Difference Equation

$$\mathbf{y}[k] = \mathbf{F}\left(\mathbf{y}[k-1], \mathbf{y}[k-2], \mathbf{y}[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3]; \boldsymbol{\Theta}\right).$$

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Output Error, Equation Error and Error-in-Variables

$$\mathbf{u}[k] = \mathbf{u}^*[k] + \mathbf{s}[k],$$
$$\mathbf{y}^*[k] = \mathbf{F}\left(\mathbf{y}^*[k-1], \mathbf{y}^*[k-2], \mathbf{y}^*[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3]\right) + \mathbf{v}[k],$$
$$\mathbf{y}[k] = \mathbf{y}^*[k] + \mathbf{w}[k].$$

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Noise Model

System Identification Procedure

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Validation Data

System Identification Procedure

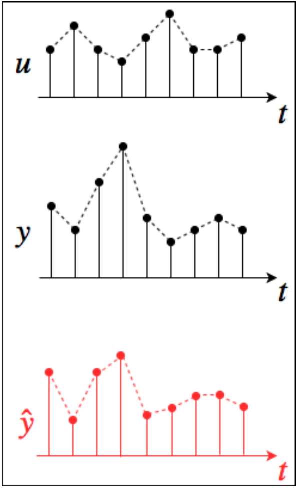

Figure: Comparison between measured data and predicted values

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One-step-ahead Prediction *vs* Free-run Simulation

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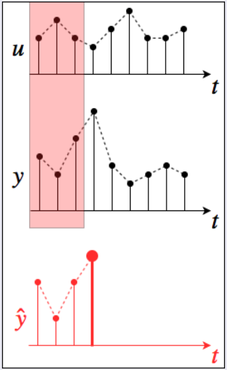
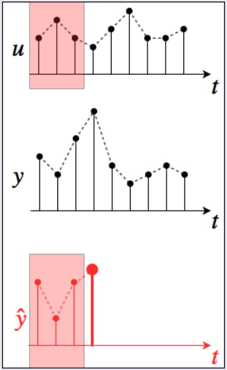
One-step-ahead Prediction



Free-run Simulation



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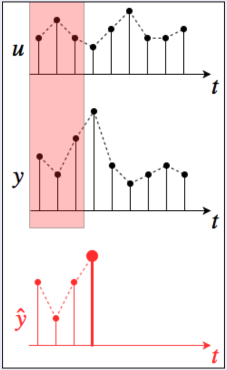

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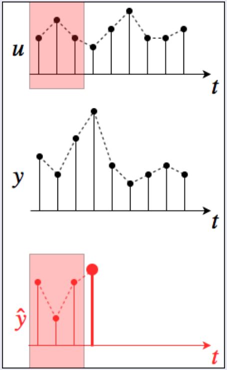

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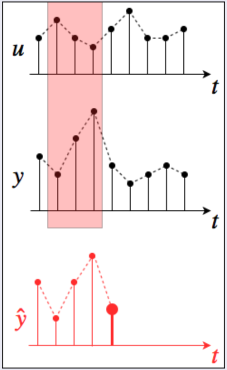

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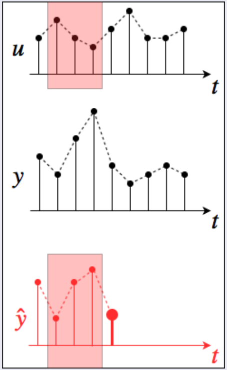

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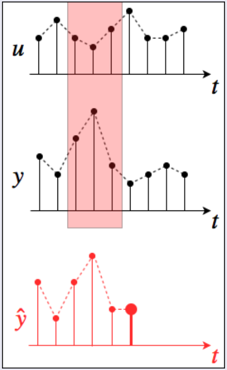

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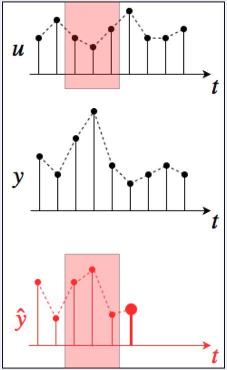

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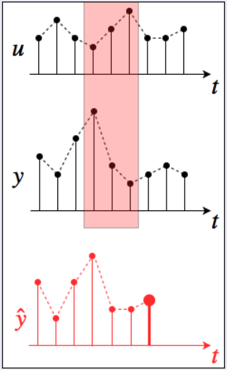

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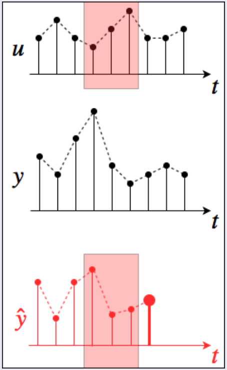

One-step-ahead Prediction *vs* Free-run Simulation

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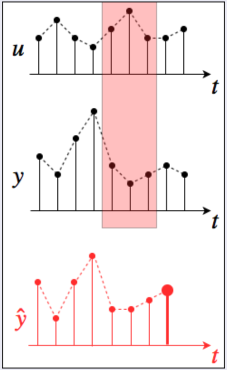

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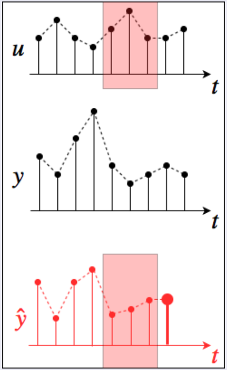

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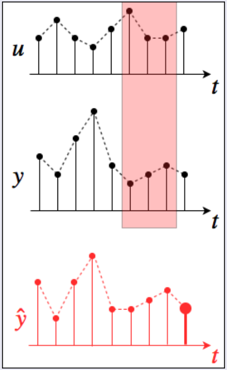

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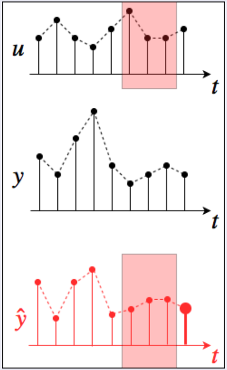
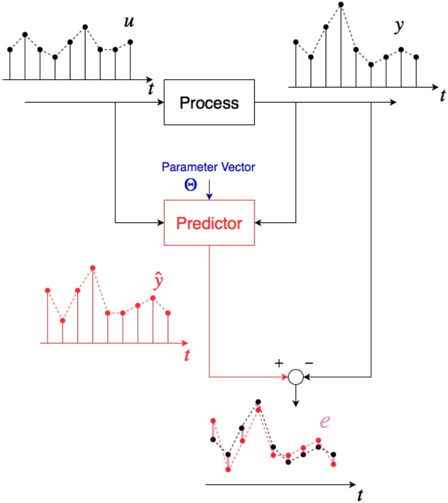




General Framework

- Noise model ⇒ Optimal Predictor:

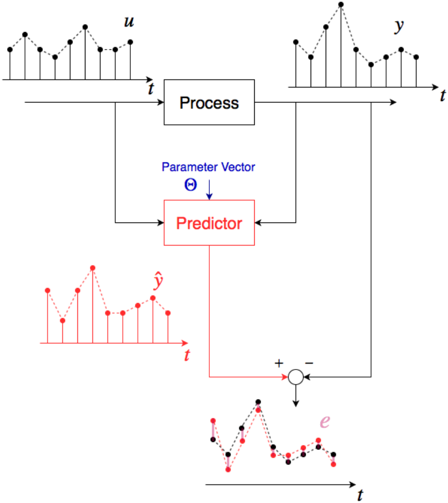
$$\hat{\mathbf{y}}[k] = E\{\mathbf{y}[k] \mid k-1\}$$

- Compute errors:

$$\mathbf{e}[k] = \hat{\mathbf{y}}[k] - \mathbf{y}[k]$$

- Find parameter Θ such the sum of square errors is minimized:

$$\min_{\Theta} \sum_k \|\mathbf{e}[k]\|^2$$



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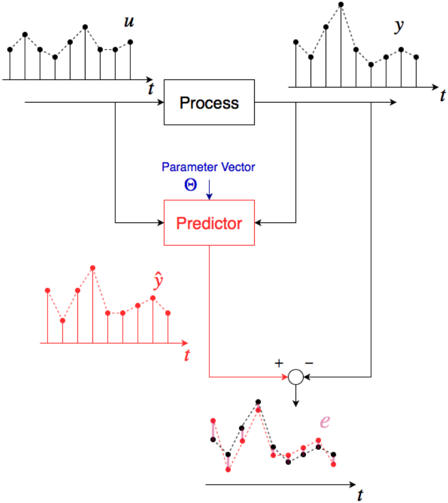

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- Find parameter $\boldsymbol{\Theta}$ such the sum of square errors is minimized:

$$\min_{\boldsymbol{\Theta}} \sum_k \|\mathbf{e}[k]\|^2$$

NARX (Nonlinear AutoRegressive with eXogenous input) model.

True system

$$\mathbf{y}[k] = \mathbf{F}(\mathbf{y}[k-1], \mathbf{y}[k-2], \mathbf{y}[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3]; \boldsymbol{\Theta}) + \underbrace{\mathbf{v}[k]}_{\text{white noise}}.$$

Optimal Predictor

One-step-ahead prediction:

$$\hat{\mathbf{y}}_1[k] = \mathbf{F}(\mathbf{y}[k-1], \mathbf{y}[k-2], \mathbf{y}[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3]; \boldsymbol{\Theta}).$$

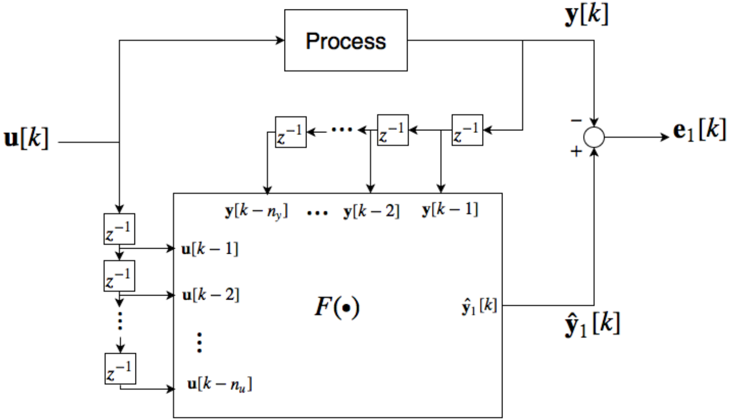

Figure: NARX model prediction error.

NOE Model

Prediction Error Methods

NOE (Nonlinear Output Error) model.

True system

$$\mathbf{y}^*[k] = \mathbf{F}\left(\mathbf{y}^*[k-1], \mathbf{y}^*[k-2], \mathbf{y}^*[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3]; \mathbf{\Theta}\right),$$
$$\mathbf{y}[k] = \mathbf{y}^*[k] + \underbrace{\mathbf{w}[k]}_{\text{white noise}} .$$

Optimal Predictor

Free-run simulation:

$$\hat{\mathbf{y}}_s[k] = \mathbf{F}(\hat{\mathbf{y}}_s[k-1], \hat{\mathbf{y}}_s[k-2], \hat{\mathbf{y}}_s[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3]; \mathbf{\Theta}).$$

NOE Model

Prediction Error Methods

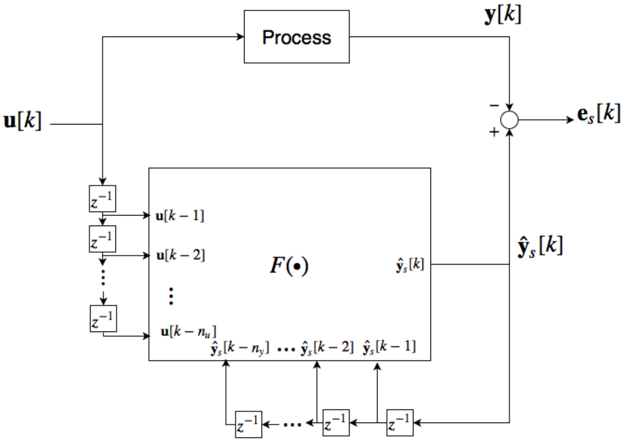

Figure: NOE model prediction error.

NARMAX Model

Prediction Error Methods

NARMAX (Nonlinear AutoRegressive Moving Average with eXogenous input) model.

True system

$$\mathbf{y}[k] = \mathbf{F}(\mathbf{y}[k-1], \mathbf{y}[k-2], \mathbf{y}[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3], \mathbf{v}[k-1], \mathbf{v}[k-2], \mathbf{v}[k-3]; \boldsymbol{\Theta}) + \underbrace{\mathbf{v}[k]}_{\text{white noise}}.$$

Optimal Predictor

$$\hat{\mathbf{y}}_v[k] = \mathbf{F}(\mathbf{y}[k-1], \mathbf{y}[k-2], \mathbf{y}[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3], \mathbf{y}[k-1] - \hat{\mathbf{y}}[k-1], \mathbf{y}[k-2] - \hat{\mathbf{y}}[k-2], \mathbf{y}[k-3] - \hat{\mathbf{y}}[k-3]; \boldsymbol{\Theta}).$$

NARMAX (Nonlinear AutoRegressive Moving Average with eXogenous input) model.

True system

$$\mathbf{y}[k] = \mathbf{F}(\mathbf{y}[k-1], \mathbf{y}[k-2], \mathbf{y}[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3], \mathbf{v}[k-1], \mathbf{v}[k-2], \mathbf{v}[k-3]; \mathbf{\Theta}) + \underbrace{\mathbf{v}[k]}_{\text{white noise}} .$$

Optimal Predictor

$$\hat{\mathbf{y}}_v[k] = \mathbf{F}(\mathbf{y}[k-1], \mathbf{y}[k-2], \mathbf{y}[k-3], \mathbf{u}[k-1], \mathbf{u}[k-2], \mathbf{u}[k-3], \mathbf{y}[k-1] - \hat{\mathbf{y}}[k-1], \mathbf{y}[k-2] - \hat{\mathbf{y}}[k-2], \mathbf{y}[k-3] - \hat{\mathbf{y}}[k-3]; \mathbf{\Theta}).$$

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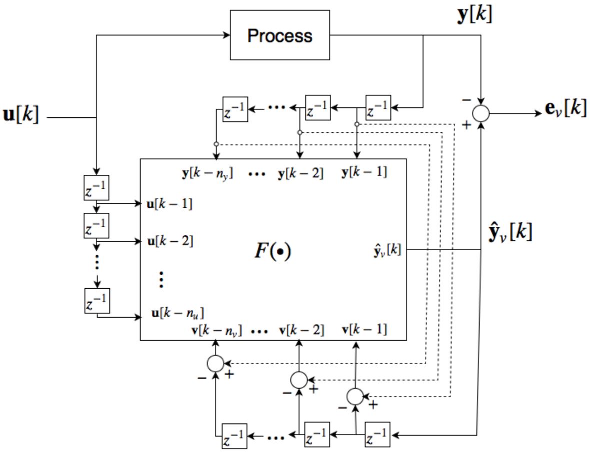

Figure: NARMAX model prediction error.

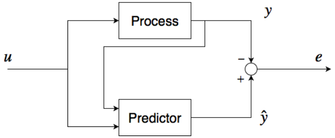

Figure: Prediction depends only on measured values.

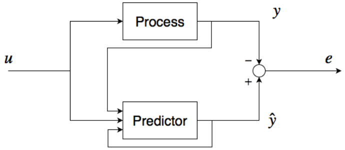

Figure: Predictor has a recurrent structure.

Chalenges

- Unboundedness;
- Multiple Minima.

Minimizing the sum of squared errors:

$$\min_{\boldsymbol{\Theta}} V(\boldsymbol{\Theta}) = \|\mathbf{e}(\boldsymbol{\Theta})\|^2$$

Derivatives:

$$\nabla V(\mathbf{\Theta}) = J(\mathbf{\Theta})^T \ \mathbf{e}(\mathbf{\Theta}),$$

$$\nabla^2 V(\mathbf{\Theta}) = J^T(\mathbf{\Theta}) J(\mathbf{\Theta}) + \sum_{i=1}^{N_e} e_i(\mathbf{\Theta}) \left(\nabla^2 e_i(\mathbf{\Theta})\right).$$

Objective Function Derivatives

Nonlinear Least Squares

Derivatives:

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- Iterative Algorithms. Starting in $\mathbf{\Theta}^0$ updates the solution:

$$\mathbf{\Theta}^{n+1} = \mathbf{\Theta}^n + \Delta\mathbf{\Theta}^n$$

- Gauss-Newton:

$$\Delta\mathbf{\Theta} = -\underbrace{\mu}_{\text{step lenght}}\Big(\underbrace{J^T(\mathbf{\Theta})J(\mathbf{\Theta})}_{\text{Hessian approx.}}\Big)^{-1}\underbrace{J(\mathbf{\Theta})^T\,\mathbf{e}(\mathbf{\Theta})}_{\text{grad.}}$$

- Levenberg-Marquardt:

$$\Delta\mathbf{\Theta} = -\Big(\underbrace{J^T(\mathbf{\Theta})J(\mathbf{\Theta})}_{\text{Hessian approx.}}+\lambda D\Big)^{-1}\underbrace{J(\mathbf{\Theta})^T\,\mathbf{e}(\mathbf{\Theta})}_{\text{grad.}}$$

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- Iterative Algorithms. Starting in $\mathbf{\Theta}^0$ updates the solution:

$$\mathbf{\Theta}^{n+1} = \mathbf{\Theta}^n + \Delta\mathbf{\Theta}^n$$

- Gauss-Newton:

$$\Delta\mathbf{\Theta} = -\underbrace{\mu}_{\text{step lenght}}\Big(\underbrace{J^T(\mathbf{\Theta})J(\mathbf{\Theta})}_{\text{Hessian approx.}}\Big)^{-1}\underbrace{J(\mathbf{\Theta})^T\ \mathbf{e}(\mathbf{\Theta})}_{\text{grad.}}$$

- Levenberg-Marquardt:

$$\Delta\mathbf{\Theta} = -\Big(\underbrace{J^T(\mathbf{\Theta})J(\mathbf{\Theta})}_{\text{Hessian approx.}} + \lambda D\Big)^{-1}\underbrace{J(\mathbf{\Theta})^T\ \mathbf{e}(\mathbf{\Theta})}_{\text{grad.}}$$

“Parallel Training Considered Harmful?”

- Parallel training $\Rightarrow$ NOE model;
- Series-parallel training $\Rightarrow$ NARX model.

Series-parallel training alleged advantages

Series-parallel to be preferred [Narendra and Parthasarathy, 1990]:

1. Bounded signals;
2. Smaller computational cost;
3. Simulated output should tend to the real one, therefore the results should not be significantly different;
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The following dynamic systems are present during the system identification procedure:

1. True System;
2. Predictor ;
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Dynamic Systems Present During Identification

Parallel Training and Unbounded Signals

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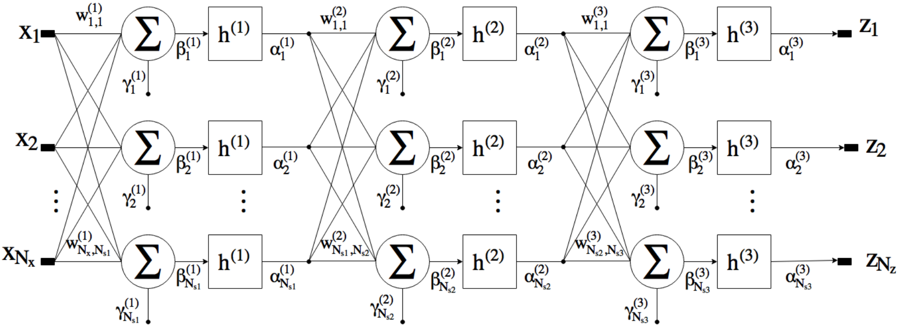
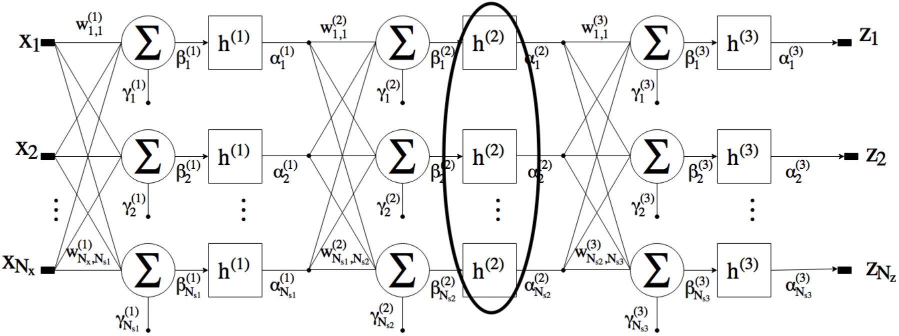


Figure: Three-layer feedforward network.



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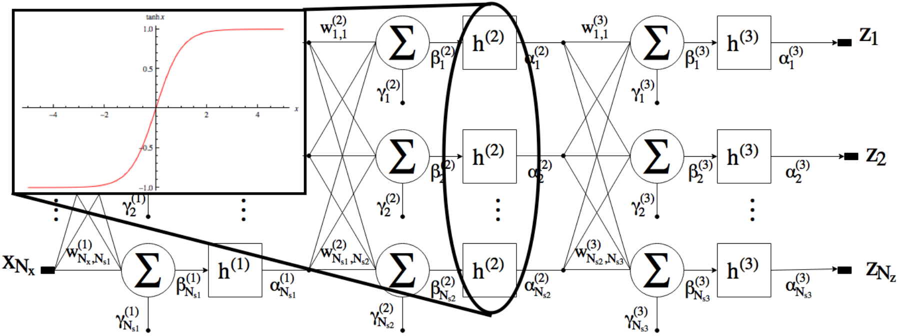
Feedforward Network

Neural Network Training



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Computational Cost per Stage

Complexity Analysis

Stage - Levenberg-Marquardt	**Series-parallel**	**Parallel**
Compute error vector **e**	$\mathcal{O}(N \cdot N_w)$	$\mathcal{O}(N \cdot N_w)$
Compute Jacobian matrix J	$\mathcal{O}(N \cdot N_w \cdot N_y)$	$\mathcal{O}(N \cdot N_\Theta \cdot N_y^2)$
Parameter update $\Delta\boldsymbol{\Theta} = -\left(J^T J + \lambda D\right)^{-1} J^T \mathbf{e}.$	$\mathcal{O}(N \cdot N_\Theta^2 + N_\Theta^3)$	$\mathcal{O}(N \cdot N_\Theta^2 + N_\Theta^3)$

Table: Complexity Analysis

Relation

$$N_y < N_y^2 < N_w \approx N_\Theta$$

Computational Cost per Stage

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Neural Network Training

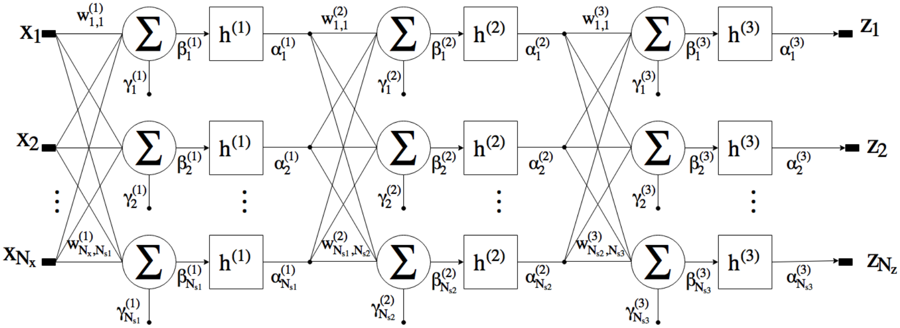

Figure: Three-layer feedforward network.

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Computer Generated Example

Comparing Parallel and Series-parallel Models

Problem Statement

- Generate data using the following system: [Chen et al., 1990]

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\begin{aligned}
y^*[k] &= (0.8 - 0.5\exp(-y^*[k-1]^2)y^*[k-1] - \\
&\quad (0.3 + 0.9\exp(-y^*[k-1]^2)y^*[k-2] + u[k-1] + \\
&\quad 0.2u[k-2] + 0.1u[k-1]u[k-2] + v[k] \\
y[k] &= y^*[k] + w[k].
\end{aligned}
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- 10 nodes in the hidden layer;
- 800 samples for identification and 200 samples for validation;
- Compare error in validation window.

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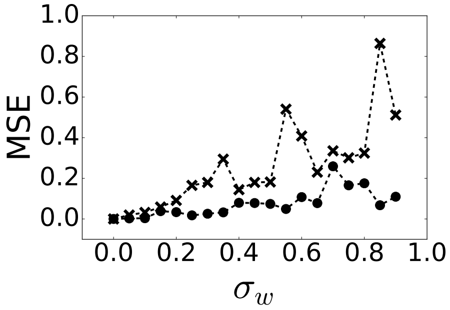

(a) $\sigma_v = 0$; (b) $\sigma_w = 0$;

Figure: MSE (mean square error) *vs* noise levels on the validation window for parallel training ($\bullet$) and series-parallel training ($\times$).

Table: Running time.

Experiment Conditions		Execution time	
N_{hidden}	N	Parallel Training	Series-parallel Training
10	1000 samples	3.7 s	3.1 s
30	1000 samples	6.4 s	5.7 s
10	5000 samples	14.6 s	11.0 s
30	5000 samples	18.5 s	17.5 s

Computer Generated Example

Comparing Parallel and Series-parallel Models

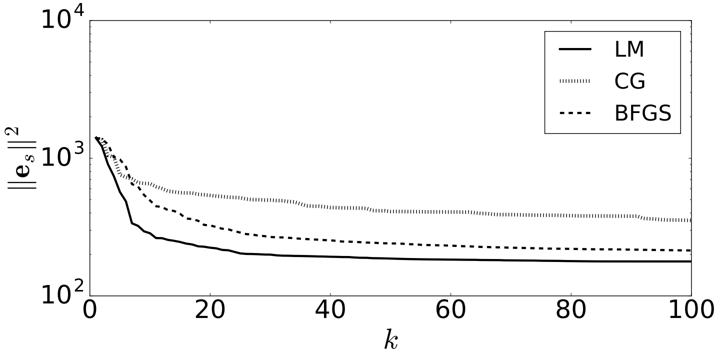

Figure: Sum of squared simulation errors per epoch for:
Levenberg-Marquardt (LM); Conjugate-gradient (CG); and, BFGS

Optimization Methods and Unboundedness

First-Order Linear System

$$\hat{y}[k] = \theta_1 \hat{y}[k-1] + \theta_2 u[k-1]$$

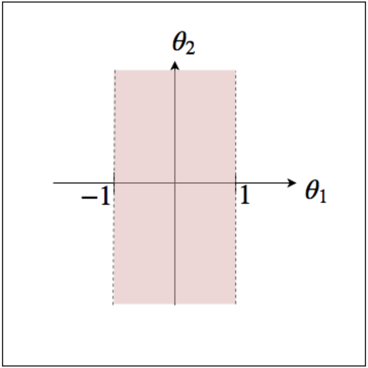

Figure: Set of parameters (θ_1, θ_2) that yield a bounded solution $\hat{y}[k]$.

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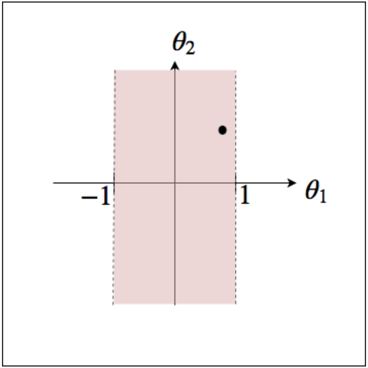

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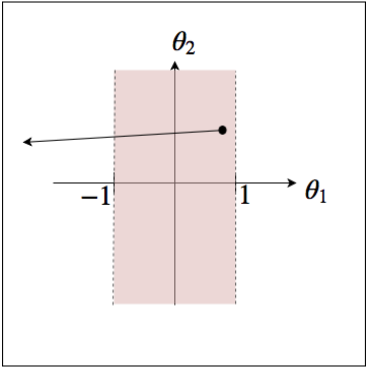

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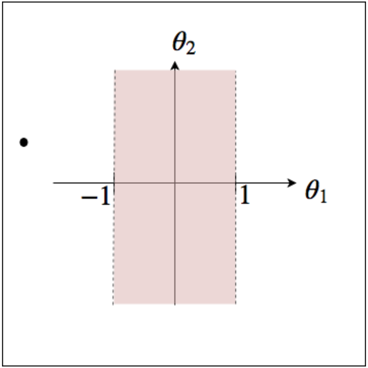

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- Trust-region methods;
- Levenberg-Marquardt;
- Backtrack line search;
- Pattern-Search;

Multiple Shooting

Multiple Shooting

- Applications:
  1. Boundary values problems;
  2. ODE parameter estimation;
  3. Optimal control;
- Escape local minima;
- Better numerical stability;
- Can be implemented in parallel.

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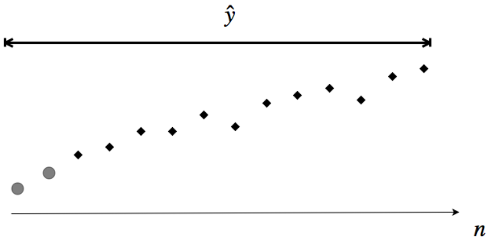


Figure: The initial conditions are represented with circles ○ and subsequent simulated values with diamonds ◇.

Single Shooting

Estimate NOE model solving:

$$\min_{\mathbf{\Theta}} \|\mathbf{e}_s\|^2$$

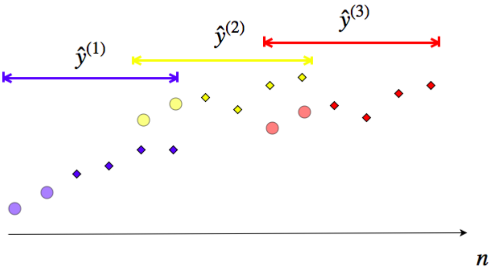


Figure: Three consecutive simulations $\hat{y}^{(i)}$, $i = 1, 2, 3$ are indicated with different colors. The initial conditions are represented with circles $\bigcirc$ and subsequent simulated values with diamonds $\diamond$.

Multiple Shooting

- m_s subdivisions.
- $\hat{y}^{(i)} \Rightarrow$ i-th simulation.
- $\mathbf{e}_s^{(i)} \Rightarrow$ i-th error.
- $\mathbf{e}_{\text{ms}} = \begin{bmatrix} \mathbf{e}_s^{(1)} \\ \vdots \\ \mathbf{e}_s^{(m_s)} \end{bmatrix}$

Multiple Shooting

Shooting Methods for Parameter Estimation of Output Error Models

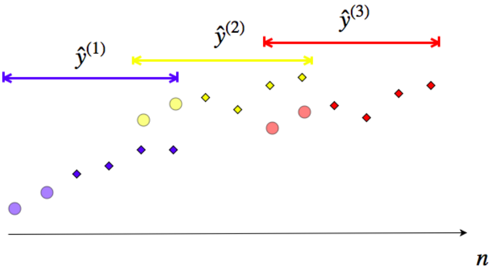
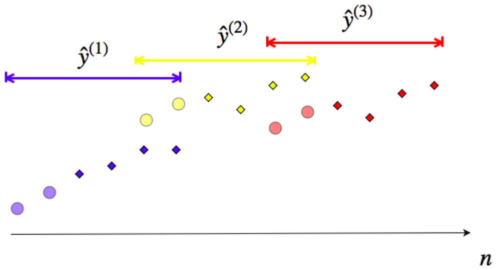


Figure: Three consecutive simulations $\hat{y}^{(i)},\ i = 1, 2, 3$ are indicated with different colors.The initial conditions are represented with circles $\bigcirc$ and subsequent simulated values with diamonds $\diamond$.

Multiple Shooting

- m_s subdivisions.
- $\hat{y}^{(i)} \Rightarrow i$-th simulation.
- $\mathbf{e}_s^{(i)} \Rightarrow i$-th error.
- $\mathbf{e}_{\text{ms}} = \begin{bmatrix} \mathbf{e}_s^{(1)} \\ \vdots \\ \mathbf{e}_s^{(m_s)} \end{bmatrix}$



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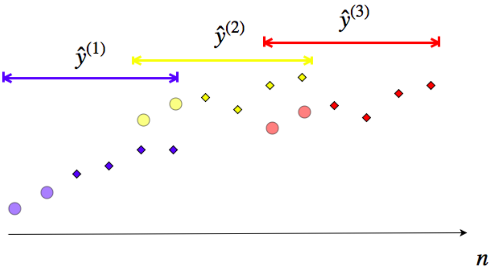

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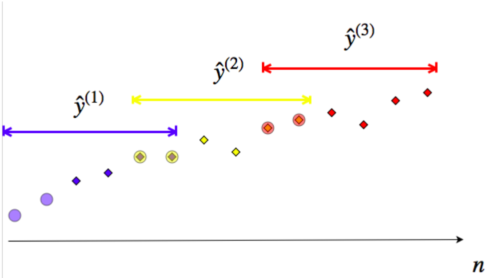

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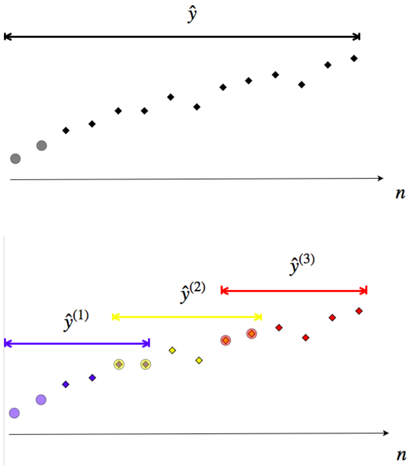
Multiple Shooting

- $\mathbf{e}_{\text{ms}} = \mathbf{e}_s$ if initial conditions matches the previous ones.



Single Shooting

Estimate NOE model solving:

$$\min_{\boldsymbol{\Theta}} \|\mathbf{e}_s\|^2$$

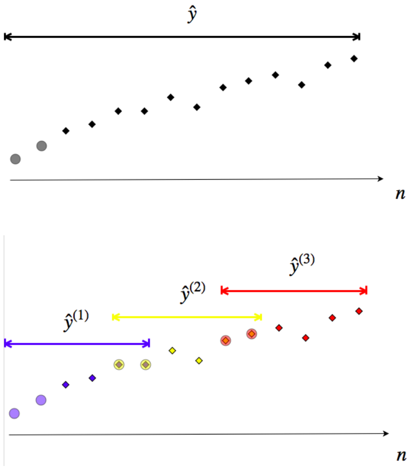
Multiple Shooting

Estimate NOE model solving:

$$\min_{\boldsymbol{\Phi}} \|\mathbf{e}_{\mathrm{ms}}\|^2$$

subject to: $\underline{\hat{\mathbf{y}}}^{(i)}[\mathbf{end}] = \underline{\mathbf{y}}_0^{(i+1)}$

$$i = 1, \cdots, m_s$$



Single Shooting

Parameter $\mathbf{\Theta}$.

Multiple Shooting

Extended parameter $\mathbf{\Phi}$:

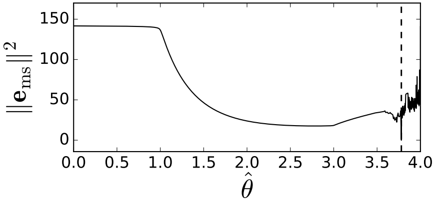
$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Theta} \\ \underline{\mathbf{y}}_0^{(1)} \\ \vdots \\ \underline{\mathbf{y}}_0^{(m_s)} \end{bmatrix}$$

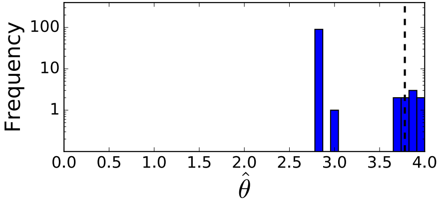
Logistic Map

A dataset with 300 samples were generated using the logistic map:

$$y[k] = \theta y[k-1](1 - y[k-1]),$$

for $\theta = 3.78$.





$$m_s = 1$$

Logistic Map

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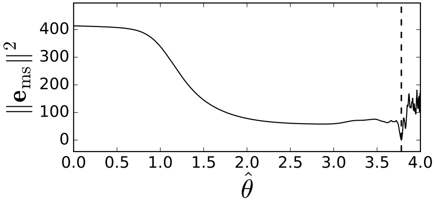

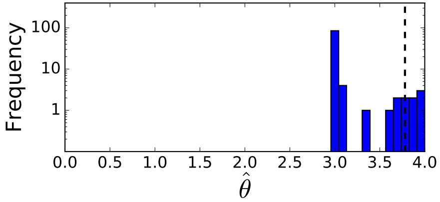

$$m_s = 30$$

Logistic Map

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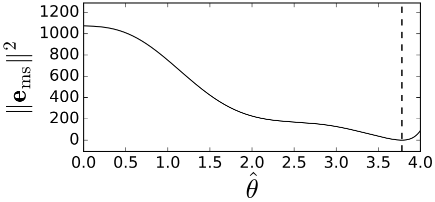

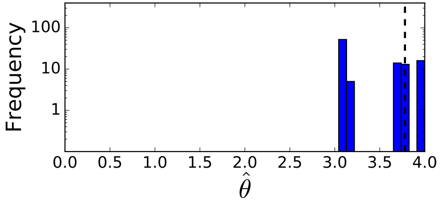

$m_s = 100$

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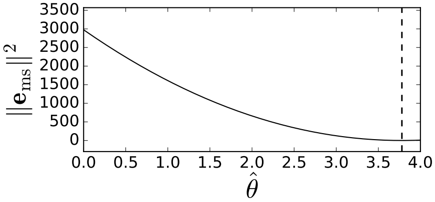

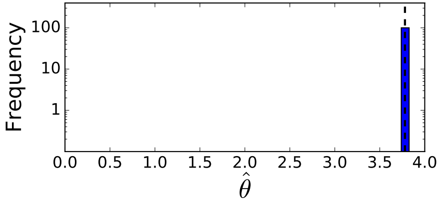

$m_s = 300$

Conclusion

Future Work

- Penalty method $\Rightarrow$ Byrd-Omojokun SQP method;
- Structure selection procedure using l^1 regularization:

$$\min_{\mathbf{\Theta}} \|\mathbf{e}\|_2^2 + \mu\|\mathbf{\Theta}\|_1$$

- And its application to multiple shooting:

$$\begin{aligned} \min_{\mathbf{\Phi}} \quad & \tfrac{1}{2}\|\mathbf{e}_{\text{ms}}\|^2 + \mu\|\mathbf{\Theta}\|_1 \\ \text{subject to:} \quad & \hat{\underline{\mathbf{y}}}^{(i)}[\text{end}] = \underline{\mathbf{y}}_0^{(i+1)}, \; i = 1, \cdots, m_s - 1. \end{aligned}$$

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The End